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A zero-free interval for chromatic polynomials of graphs with 3-leaf spanning trees

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Abstract

It is proved that if G is a graph containing a spanning tree with at most three leaves, then the chromatic polynomial of G has no roots in the interval $(1, t_1]$, where $t_1 \approx 1.2904$ is the smallest real root of the polynomial $(t-2)^6 + 4(t-1)^2(t-2)^3 - (t-1)^4$. We also construct a family of graphs containing such spanning trees with chromatic roots converging to t_1 from above. We employ the Whitney 2-switch operation to manage the analysis of an infinite class of chromatic polynomials.

Keywords: 05C31; chromatic polynomial; zero-free interval; spanning tree; splitting-closed

1 Introduction

The *chromatic polynomial* $P(G, t)$ of a graph G is a polynomial with integer coefficients which counts for each non-negative integer t, the number of t-colourings of G. It was introduced by Birkhoff [2] in 1912 for planar graphs, and extended to all graphs by Whitney [9, 10] in 1932. If t is a real number then we say t is a *chromatic root* of G if $P(G, t) = 0$. Thus the numbers $0, 1, 2, \dots, \chi(G) - 1$ are always chromatic roots of G and, in fact, the only rational ones. On the other hand, it is easy to see that the interval $(-\infty, 0)$ contains no chromatic roots, and Tutte [8] showed that the same is true for the interval $(0, 1)$. We say that such intervals are *zero-free* for the class of all graphs. In 1993, Jackson [5] proved the surprising result that the interval $(1, 32/27]$ is also zero-free, and found a sequence of graphs whose chromatic roots converge to $32/27$ from above. Thomassen [6] strengthened this by showing that the set of chromatic roots consists of 0, 1, and a dense subset of the interval $(32/27, \infty)$.

Let $Q(G, t) = (-1)^{|V(G)|} P(G, t)$, and $b(G)$ be the number of blocks of G. We say that G is *separable* if $b(G) \geq 2$ and *non-separable* otherwise. Note that K_2 is non-separable.

In [7], Thomassen provided a new link between Hamiltonian paths and colourings by proving that the zero-free interval of Jackson can be extended when G has a Hamiltonian path. More precisely he proved the following.

Theorem 1.1. [7] *If G is a non-separable graph with a Hamiltonian path, then $Q(G,t) > 0$ for $t \in (1, t_0]$, where $t_0 \approx 1.295$ is the unique real root of the polynomial $(t-2)^3+4(t-1)^2$. Furthermore, for all $\varepsilon > 0$ there exists a non-separable graph with a Hamiltonian path whose chromatic polynomial has a root in the interval $(t_0, t_0 + \varepsilon)$.*

If G is separable and has a Hamiltonian path, then it is easily seen using Theorem 1.1 and Proposition 2.2 that $Q(G,t)$ is non-zero in the interval $(1, t_0]$ with sign $(-1)^{b(G)-1}$.

For a graph G, a *k-leaf spanning tree* is a spanning tree of G with at most k leaves (vertices of degree 1). We denote the class of non-separable graphs which admit a k-leaf spanning tree by $\mathcal{G}_k$. Thus, Theorem 1.1 gives a zero-free interval for the class $\mathcal{G}_2$. In this article we prove the following analogous result for the class $\mathcal{G}_3$.

Theorem 1.2. *If G is a non-separable graph with a 3-leaf spanning tree, then $Q(G,t) > 0$ for $t \in (1, t_1]$, where $t_1 \approx 1.2904$ is the smallest real root of the polynomial $(t-2)^6+4(t-1)^2(t-2)^3-(t-1)^4$. Furthermore, for all $\varepsilon > 0$, there exists a non-separable graph with a 3-leaf spanning tree whose chromatic polynomial has a root in the interval $(t_1, t_1 + \varepsilon)$.*

A natural extension of this work would be to find $\varepsilon_k > 0$ so that $(1, 32/27 + \varepsilon_k]$ is zero-free for the class $\mathcal{G}_k$, $k \geq 4$. However since the graphs presented by Jackson in [5] are non-separable, it must be that $\varepsilon_k \to 0$ as $k \to \infty$. Another possible extension would be to find $\varepsilon_\ell > 0$ so that $(1, 32/27 + \varepsilon_\ell]$ is zero-free for the family of graphs containing a spanning tree T with $\Delta(T) \leq 3$ and at most ℓ vertices of degree 3. Here the possible implications are much more interesting since it is not clear if $\varepsilon_\ell \to 0$ as $\ell \to \infty$. Indeed a short argument shows that only finitely many of the graphs presented by Jackson in [5] have a spanning tree of maximum degree 3. Theorem 1.1 and our result solve the cases $\ell = 0$ and $\ell = 1$ respectively, which leads us to conjecture the following.

Conjecture 1.1. *There exists $\varepsilon > 0$ such that if G is a non-separable graph with a spanning tree of maximum degree 3, then $Q(G,t) > 0$ for $t \in (1, 32/27 + \varepsilon]$.*

Barnette [1] proved that a 3-connected planar graph has a spanning tree of maximum degree 3. Thus an affirmative answer to Conjecture 1.1 would immediately imply a zero-free interval for the class of 3-connected planar graphs. Such an interval was found by Dong and Jackson [3] but it is thought to be far from maximal.

2 Preliminaries

All graphs in this article are *simple*, that is they have no loops or multiple edges. If u and v are vertices of G, then G/uv denotes the graph obtained by deleting the edge uv if it exists, and then identifying the vertices u and v. This operation is referred to as the *contraction* of uv. If G is connected, $S \subset V(G)$, and $G - S$ is disconnected, then S is called a *cut-set* of G. A *2-cut* of G is a cut-set S with $|S| = 2$. If S is a cut-set of G

and C is a component of $G - S$, then we say the graph $G[V(C) \cup S]$ is an *S-bridge* of G. Finally, if P is a path and $x, y \in V(P)$ then $P[x, y]$ denotes the subpath of P from x to y.

We make repeated use of two fundamental results in the study of chromatic polynomials.

Proposition 2.1 (Deletion-contraction identity)**.** *If G is a graph and uv is an edge of G, then*

$$P(G,t) = P(G - uv, t) - P(G/uv, t).$$

Proposition 2.2 (Factoring over complete subgraphs)**.** *If $G = G_1 \cup G_2$ be a graph such that $G[V(G_1) \cap V(G_2)]$ is a complete graph on r vertices, then*

$$P(G,t) = \frac{P(G_1,t)P(G_2,t)}{P(K_r,t)}.$$

The next proposition is easily proven using Propositions 2.1 and 2.2. The operation involved is often called a *Whitney 2-switch.*

Proposition 2.3. *Let G be a graph and $\{x, y\}$ be a 2-cut of G. Let C denote a connected component of $G - \{x, y\}$. Define G' to be the graph obtained from the disjoint union of $G - C$ and C by adding for all $z \in V(C)$ the edge xz (respectively yz) if and only if yz (respectively xz) is an edge of G. Then we have $P(G,t) = P(G',t)$.*

If G' can be obtained from G by a sequence of Whitney 2-switches, then $P(G,t) = P(G',t)$ and we say G and G' are *Whitney equivalent.*

Definition 2.1. *A graph G is a generalised triangle if the following conditions hold:*

- *G is non-separable but not 3-connected.*
- *For every 2-cut $\{x, y\}$, $xy \notin E(G)$ and G has precisely three $\{x, y\}$-bridges, all of which are separable.*

The class of generalised triangles was first defined by Jackson in [5] and is an important class of graphs in the study of chromatic roots. The name is derived from an equivalent characterisation, which says that the generalised triangles are the graphs that can be obtained from K_3 by repeatedly replacing an edge uv by two paths of length 2 with ends u and v, see Jackson [5].

2.1 Hamiltonian Paths

We briefly describe a number of results, quantities, and definitions from Thomassen [7], as they will play an important role in our result.

For each natural number $k \geq 1$, let H_k denote the graph obtained from a path $x_1x_2 \ldots x_{2k+3}$ by adding the edges x_1x_4, $x_{2k}x_{2k+3}$, and all edges x_ix_{i+4} for $i \in \{2, 4, \ldots, 2k-2\}$. Figure 1 shows the graph H_3. Also define $H_0 = K_3$ and let $\mathcal{H} = \{H_i : i \in \mathbb{N}_0\}$. Finally, let R denote the set of chromatic roots of all $H_i \in \mathcal{H}$. Thomassen [7] showed

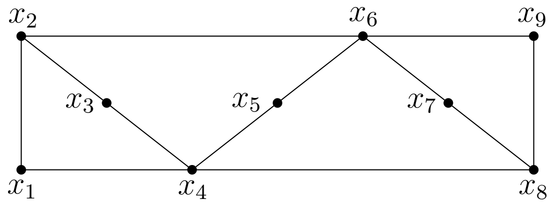

Figure 1: The graph H_3.

that a smallest counterexample to Theorem 1.1 is a member of $\mathcal{H}$. He also showed that t_0 is the infimum of $R \setminus \{0,1\}$, where t_0 is the unique real root of the polynomial $(t-2)^3+4(t-1)^2$. Together these results imply Theorem 1.1. Thomassen [7] also showed that for fixed $t \in (1, t_0]$, the value of the chromatic polynomial of H_k at t can be expressed as

$$P(H_k, t) = A\alpha^k + B\beta^k \tag{1}$$

where A, B, α and β are constants depending on t, defined by the following relations [7].

$$\delta = \sqrt{(t-2)^4 + 4(t-1)^2(t-2)} \tag{2}$$

$$\alpha = \frac{1}{2}((t-2)^2 + \delta), \quad \beta = \frac{1}{2}((t-2)^2 - \delta) \tag{3}$$

$$A + B = t(t-1)(t-2) \tag{4}$$

$$A\alpha + B\beta = t(t-1)((t-2)^3 + (t-1)^2) \tag{5}$$

Noting that $\alpha + \beta = (t-2)^2$, we have more explicitly

$$A = \frac{1}{\delta}t(t-1)((t-2)\alpha + (t-1)^2)$$

$$B = t(t-1)(t-2) - A.$$

As stated in [7], it can be routinely verified that $0 < \beta < \alpha < 1$ and $0 < B < -A < 1$ for $t \in (1, t_0)$.

3 A Special Class of Graphs with 3-leaf Spanning Trees

Let $F_k = H_k - x_1x_2$. If G is a graph, $\{x, y\}$ is a 2-cut of G and B is an $\{x, y\}$-bridge, then we write $B = F(x, y, k)$ to indicate that B is isomorphic to F_k, where x is identified with x_1 and y is identified with x_2 in G. For $i, j, k \in \mathbb{N}_0$, define $G_{i,j,k}$ to be the graph composed of two vertices x and y, and three $\{x, y\}$-bridges $F(x, y, i)$, $F(x, y, j)$ and $F(y, x, k)$. Figure 2 shows the graph $G_{4,2,3}$. Note that if $j = 0$, then $G_{i,j,k}$ is isomorphic to H_{i+k+2}.

Lemma 3.1 characterises the bridges of a generalised triangle with a Hamiltonian path. It can be found in the work of Thomassen [7] and will be useful for us as a lemma.

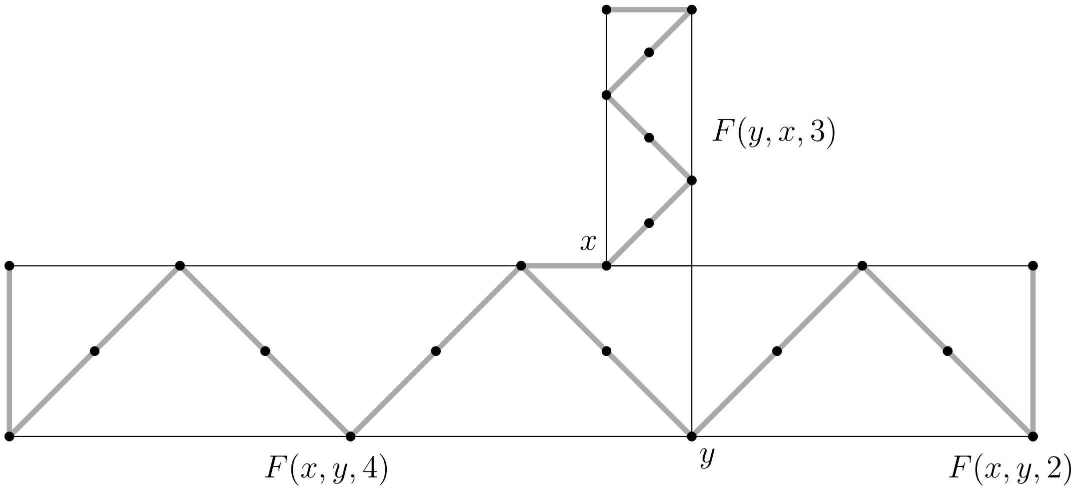

Figure 2: The graph $G_{4,2,3}$ and a 3-leaf spanning tree thereof.

Lemma 3.1. *Let G be a generalised triangle, $\{x, y\}$ be a 2-cut of G and B be an $\{x, y\}$-bridge of G.*

(a) If B contains a Hamiltonian path P starting at x and ending at y, then $B = F(x, y, 0)$, i.e. B is a path of length 2.

(b) If B contains a path P starting at y and covering all vertices of B except for x, then $B = F(x, y, k)$ for some $k \in \mathbb{N}_0$.

Proof. (a) Since G is a generalised triangle, B is separable and has a cut-vertex v. The Hamiltonian path P of B shows that neither of $G - \{x, v\}$ and $G - \{y, v\}$ can have more than two components. Thus, since G is a generalised triangle, neither $\{x, v\}$ nor $\{y, v\}$ is a cut-set of G and so $|V(B)| = 3$. Since B is connected and $xy \notin E(G)$, B is a path of length 2 as claimed.

(b) Again B is separable and has a cut-vertex v. We proceed by induction on $|V(G)|$. If $|V(B)| = 3$ then $B = F(x, y, 0)$, so we may assume that $|V(B)| \geq 4$ and the result is true for all bridges on fewer vertices. At least one of $\{x, v\}$ or $\{y, v\}$ is a 2-cut of G, but $\{x, v\}$ cannot be since $G - \{x, v\}$ has at most two components. Thus v is the unique neighbour of x in B, and $\{y, v\}$ is a 2-cut of G with precisely three $\{y, v\}$-bridges, two of which, say B_1 and B_2, are contained in B. Suppose without loss of generality that B_1 contains the subpath of P from x to v. Then $P[V(B_1)]$ is a Hamiltonian path of B_1 and so by part (a), $B_1 = F(y, v, 0)$. $P[V(B_2)]$ is a path in B_2, starting at v and covering all vertices of B_2 except for y. By induction, $B_2 = F(y, v, k - 1)$ for some $k - 1 \in \mathbb{N}_0$. It follows that $B = F(x, y, k)$.

□

It is easy to see that each $G_{i,j,k}$ is a generalised triangle and contains a 3-leaf spanning tree. The following result shows that it is enough to only consider these graphs.

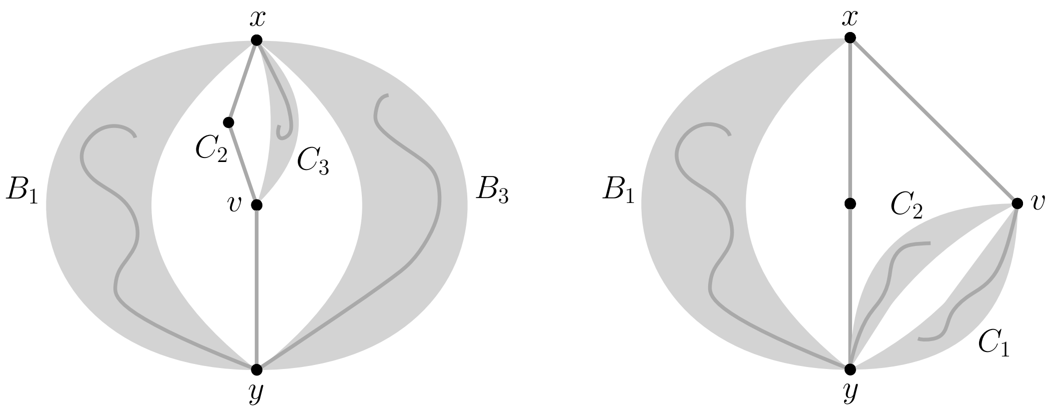

Figure 3: Cases 1 and 2 in the proof of Lemma 3.2.

Lemma 3.2. *Let G be a graph with a 3-leaf spanning tree T. If G is a generalised triangle, then $P(G,t) = P(G_{i,j,k},t)$ for some $i,j,k \in \mathbb{N}_0$.*

Proof. We show that G is Whitney equivalent to $G_{i,j,k}$ for some $i,j,k \in \mathbb{N}_0$. By the remark following Proposition 2.3, this implies the result. If G contains a Hamiltonian path then for any 2-cut $\{x,y\}$, one of the three $\{x,y\}$-bridges satisfies Lemma 3.1(a), whilst the other two satisfy Lemma 3.1(b). Thus G is Whitney equivalent to $G_{i,0,k}$ for some $i,k \in \mathbb{N}_0$ and we are done.

Now we may assume that v is a vertex of degree 3 in T. We first find a useful 2-cut. Since G is not 3-connected, there is some cut set of size 2. Choose such a 2-cut $S = \{x,y\}$ so that the smallest S-bridge containing v is as small as possible. We claim that $v \in S$. If this is not the case then let B be the S-bridge containing v. Since B is separable there is some cut-vertex u of B. Also, since v has degree 3 in T, $|V(B)| \geq 4$. This implies that one of $\{x,u\}$ or $\{y,u\}$ is a smaller cut-set containing v, a contradiction. We now claim to be able to find a 2-cut S such that $v \in S$, and the three neighbours of v in T, denoted v_1, v_2 and v_3, lie in three different S-bridges. If this is not already the case, choose a 2-cut S such that $v \in S$, and the S-bridge containing two of v_1, v_2, v_3 is as small as possible. By a similar argument we find a 2-cut with the desired property.

Fix the 2-cut $S = \{x,y\}$ so that y has degree 3 in T. For $i \in \{1,2,3\}$, let B_i be an S-bridge, and $y_i \in V(B_i)$ be the neighbours of y in T. Finally for $i \in \{1,2,3\}$ we let P_i be the unique path in T from y to a leaf of T, which contains the vertex y_i. Suppose without loss of generality that x lies on P_2. We distinguish two cases.

Case 1: $V(P_2) = V(B_2)$.
P_1 and P_3 are paths in B_1 and B_3 respectively, which start at y and cover all vertices of B_1, B_3 except x. By Lemma 3.1(b), $B_1 = F(x,y,i)$ and $B_3 = F(x,y,k)$ for some $i,k \in \mathbb{N}_0$. If P_2 ends at x then Lemma 3.1(a) implies $B_2 = F(x,y,0)$ and by performing a Whitney 2-switch of B_3 about $\{x,y\}$ we are done. So suppose P_2 ends at some vertex z other than x. B_2 is separable and so there is a cut-vertex v of B_2. Since P contains a subpath connecting y and x, it follows that v lies between y and x on P. So let Q_1, Q_2 and Q_3 be the sub-paths of P_2 from y to v, v to x and x to z respectively. Now $G - \{y,v\}$ contains at most two components, and therefore Q_1 is the edge yv. Note that

$|V(B)| \geq 4$, so $G - \{x, v\}$ contains at least two components. Since G is a generalised triangle, it has precisely three $\{x, v\}$-bridges, two of which are contained in B_2. Let C_2 and C_3 be the $\{x, v\}$-bridges containing Q_2 and Q_3 respectively. Q_2 is a Hamiltonian path of C_2 from v to x. Thus by Lemma 3.1(a), $C_2 = F(x, v, 0)$. Similarly C_3 contains a path starting at x and covering all vertices of C_3 except for v. By Lemma 3.1(b) it follows that $B_3 = F(v, x, j-1)$ for some $j \in \mathbb{N}_0$. Now performing a Whitney 2-switch of C_3 with respect to $\{x, v\}$ gives a new graph, where the $\{x, y\}$-bridge corresponding to B_2 is $F(x, y, j)$. Finally, performing a Whitney 2-switch of B_3 about $\{x, y\}$ gives a graph isomorphic to $G_{i,j,k}$. This completes the proof.

Case 2: $V(P_2) \supset V(B_2)$**.**
Suppose without loss of generality that P_2 also contains vertices of $B_3 - x - y$. As before, Lemma 3.1(b) implies that $B_1 = F(x, y, i-1)$ for some $i \in \mathbb{N}_0$ and Lemma 3.1(a) implies that $B_2 = F(x, y, 0)$. Now $T[V(B_3)]$ consists of two disjoint paths Q_1 and Q_2, starting at x and y respectively. Since B_3 is separable, it has a cut-vertex v. Suppose without loss of generality that $v \in V(Q_1)$. If this is not the case we perform a Whitney 2-switch of B_3 with respect to $\{x, y\}$ and proceed similarly. Both Q_1 and Q_2 contain at least one edge, thus $|V(B_3)| \geq 4$ and at least one of $\{x, v\}$ and $\{y, v\}$ is a 2-cut of G. Suppose for a contradiction that $\{x, v\}$ is a 2-cut. $G - \{x, v\}$ has at least two components and so G has precisely three $\{x, v\}$-bridges, two of which are contained in B_3. Now $G - \{v, y\}$ can have at most two components and as such v is the unique neighbour of y in B_3. But $v \in V(Q_1)$ and therefore Q_2 is a single vertex. This contradicts the fact that Q_2 contains at least one edge.

So we may assume that $\{y, v\}$ is a 2-cut of G, and v is the unique neighbour of x in B_3. Now, as before, G has precisely three $\{v, y\}$-bridges, two of which are contained in B_3. Denote the two $\{v, y\}$-bridges which are contained in B_3 by C_1 and C_2, where Q_2 is contained in C_2. It is now easy to see that Q_2 is a path of C_2 starting at y and containing all vertices of C_2 except for v. Similarly, $Q_1[V(C_1)]$ is a path of C_1 starting at v and containing all vertices of C_1 except for y. Thus by Lemma 3.1(b), $C_1 = F(y, v, k)$ and $C_2 = F(v, y, j)$ for some $j, k \in \mathbb{N}_0$.

Now consider the 2-cut $\{v, y\}$ of G. It has three bridges two of which are C_1 and C_2 found in B_3. The third $\{v, y\}$-bridge, denoted C_3, is composed of the edge xv and the two $\{x, y\}$-bridges $F(x, y, 0)$ and $F(x, y, i-1)$ of G. By performing a Whitney 2-switch of $F(x, y, i-1)$ at $\{x, y\}$, we get a new graph G', where the $\{v, y\}$-bridge of G' corresponding to C_3 is precisely $F(v, y, i)$. Now G' is isomorphic to $G_{i,j,k}$ and so $P(G, t) = P(G_{i,j,k}, t)$. This completes the proof. $\square$

3.1 A Zero Free Interval for $P(G_{i,j,k}, t)$

We now determine the behaviour of the chromatic roots of each $G_{i,j,k}$. Here t_0 is the real number defined in Theorem 1.1 and $\mathcal{H} = \{H_i : i \in \mathbb{N}_0\}$ is the family of graphs defined in Section 2.1. It is easily seen that $G_{i,0,k} = H_{i+k+2}$. If $j = 1$, then by Proposition 2.1

and 2.2 applied to y and the cut vertex v of $F(x, y, j)$ we find that

$$\begin{aligned}P(G_{i,1,k}, t) &= P(G_{i,1,k} + vy, t) + P(G_{i,1,k}/vy, t)\\ &= (t-2)^2 P(H_{i+k+2}, t) + \frac{(t-1)}{t} P(H_{i+1}, t) P(H_k, t).\end{aligned}$$

Finally if $j \geq 2$ then using Proposition 2.1 and 2.2 on the vertices x_{2j} and x_{2j+2} of $F(x, y, j)$ gives the recurrence

$$\begin{aligned}P(G_{i,j,k}, t) &= P(G_{i,j,k} + x_{2j}x_{2j+2}, t) + P(G_{i,j,k}/x_{2j}x_{2j+2}, t)\\ &= (t-2)^2 P(G_{i,j-1,k}, t) + (t-1)^2(t-2) P(G_{i,j-2,k}, t).\end{aligned}$$

Solving this second order recurrence explicitly for fixed $t \in (1, t_0]$ gives a solution of the form $P(G_{i,j,k}, t) = C\alpha^j + D\beta^j$ where C and D are constants depending on i, k and t. Recall that α and β are defined in (3). The initial conditions corresponding to $j = 0, 1$ are

$$C + D = P(H_{i+k+2}, t), \text{ and} \tag{6}$$

$$C\alpha + D\beta = (t-2)^2 P(H_{i+k+2}, t) + \frac{t-1}{t} P(H_{i+1}, t) P(H_k, t). \tag{7}$$

Multiplying (6) by β and subtracting the resulting equation from (7) gives

$$\begin{aligned}C(\alpha - \beta) &= ((t-2)^2 - \beta) P(H_{i+k+2}, t) + \frac{t-1}{t} P(H_{i+1}, t) P(H_k, t)\\ &= \alpha P(H_{i+k+2}, t) + \frac{t-1}{t} P(H_{i+1}, t) P(H_k, t). \qquad (8)\end{aligned}$$

For convenience we define $\gamma = \gamma(t) = \alpha t/(t-1) > 0$, for $t \in (1, t_0]$. Let t_1 be the smallest real root of the polynomial $(t-2)^6 + 4(t-1)^2(t-2)^3 - (t-1)^4$. We claim that for $t \in (1, t_1]$, we have

$$\gamma\beta < -A \leq \gamma\alpha. \tag{9}$$

The left inequality follows since $\gamma\beta = -t(t-1)(t-2)$ and so by (4), $-A - \gamma\beta = B > 0$. The right side follows since for $t \in (1, t_0]$

$$\begin{array}{rrcl} & -A & \leq & \gamma\alpha \\ \Longleftrightarrow & -\frac{1}{\delta}t(t-1)((t-2)\alpha + (t-1)^2) & \leq & \frac{t}{t-1}(t-2)((t-2)\alpha + (t-1)^2) \\ \Longleftrightarrow & -(t-1)^2 & \geq & (t-2)\delta \\ \Longleftrightarrow & (t-1)^4 & \leq & (t-2)^2\delta^2. \end{array}$$

Using (2), the final inequality is seen to be satisfied when the aforementioned polynomial is non-negative.

Since each $H \in \mathcal{H}$ has an odd number of vertices, Theorem 1.1 implies $P(H, t) < 0$ for $t \in (1, t_0]$. It now follows that (8), and hence C, are negative if

$$\gamma |P(H_{i+j+2}, t)| > P(H_{i+1}, t) P(H_k, t).$$

Indeed for $t \in (1, t_1]$, we have that $0 < \beta < \alpha < 1$ and $0 < B < -A < 1$, which together with (1) and (9) implies

$$\begin{aligned} P(H_{i+1}, t)P(H_k, t) &= (A\alpha^{i+1} + B\beta^{i+1})(A\alpha^k + B\beta^k) \\ &= A^2\alpha^{i+k+1} + AB\alpha^{i+1}\beta^k + AB\alpha^k\beta^{i+1} + B^2\beta^{i+k+1} \\ &< -A\gamma\alpha^{i+k+2} - B^2\beta^{i+k+1} - B\gamma\beta^{i+k+2} + B^2\beta^{i+k+1} \\ &= \gamma(-A\alpha^{i+k+2} - B\beta^{i+k+2}) = \gamma|P(H_{i+k+2}, t)|. \end{aligned}$$

Since $C < 0$, (6) implies $D < -C$. Finally, since $\alpha > \beta$, we may conclude that $P(G_{i,j,k}, t) < 0$ for $t \in (1, t_1]$.

Now suppose that $t \in (t_1, t_0)$ is fixed. Then $-A > \gamma\alpha$. Setting $i+1 = k$ for simplicity we see that

$$\frac{P(H_k, t)^2}{\gamma|P(H_{2k+1}, t)|} = \frac{A^2\alpha^{2k} + 2AB\alpha^k\beta^k + B^2\beta^{2k}}{\gamma(-A\alpha^{2k+1} - B\beta^{2k+1})} \longrightarrow \frac{A^2}{-\gamma A\alpha} = \frac{-A}{\gamma\alpha} > 1.$$

as $k \to \infty$. Thus, for large enough i and k, $\gamma|P(H_{i+k+2}, t)| < P(H_{i+1}, t)P(H_k, t)$ and hence C is positive. Though (6) implies that D is negative, since $\alpha > \beta$ it follows that for large enough j, $P(G_{i,j,k}, t) > 0$. Since we have proven that $P(G_{i,j,k}, t) < 0$ on $(1, t_1]$, we may conclude by continuity that $P(G_{i,j,k}, t)$ has a root in (t_1, t).

4 Main Result

To prove Theorem 1.2 we shall show that a smallest counterexample is a generalised triangle. In [4], Dong and Koh extracted the essence of the proofs of Jackson [5] and Thomassen [7] and gave a general method to do this. An important part of that method is the following definition and lemma.

Definition 4.1. *A family of graphs $\mathcal{G}$ is called splitting-closed if the following conditions hold for each $G \in \mathcal{G}$.*

- *For every complete cut-set C with $|C| \leq 2$, all C-bridges of G are in $\mathcal{G}$.*
- *For every 2-cut $\{x, y\}$ such that $xy \notin E(G)$, and every $\{x, y\}$-bridge B, the graphs $B + xy$ and B/xy are in $\mathcal{G}$.*

It is straightforward to check that the family of non-separable graphs with a 3-leaf spanning tree is splitting-closed. In fact $\mathcal{G}_k$ is splitting-closed for all $k \geq 2$.

Lemma 4.1. *(Adapted from [4]) Let G be a non-separable graph and $\{x, y\}$ be a 2-cut of G with $\{x, y\}$-bridges $B_1, \ldots, B_m$ where m is odd. For fixed $i, j \in [m]$, let $B_{i,j}$ be the graph formed from $B_i \cup B_j$ by adding a new vertex w and the edges xw and wy. Let $B_\cup = \cup_{k \in [m] \setminus \{i,j\}} B_k$. For fixed $t \in (1, 2)$, if $Q(B_\cup, t)$, $Q(B_{i,j}, t)$, $Q(B_k + xy, t)$, $Q(B_k/xy, t)$ are positive for each $k \in [m] \setminus \{i, j\}$, then $Q(G, t) > 0$.*

The proof of Lemma 4.1 is a straightforward modification of the proof of Lemma 2.5 in [4]. We now prove the main theorem.

Proof of Theorem 1.2. Let G be a smallest counterexample to the theorem and let $t \in (1, t_1]$ such that $Q(G,t) \leq 0$. We show that G is a generalised triangle and hence by Lemma 3.2, $P(G,t) = P(G_{i,j,k}, t)$ for some $i, j, k \in \mathbb{N}_0$. Since no $G_{i,j,k}$ has a root less than or equal to t_1, a contradiction ensues.

By the hypotheses, G is non-separable.

Claim 1: G is not 3-connected.

If G is 3-connected then $G - e$ and G/e are non-separable for every edge $e \in E(G)$. So let v be a leaf of T, and e be an edge incident to v but not in T. Then also $G - e$ and G/e have a 3-leaf spanning tree. Since G is a smallest counterexample, $Q(G - e, t) > 0$ and $Q(G/e, t) > 0$. But now, by Proposition 2.1, it follows that $Q(G,t) = Q(G - e, t) + Q(G/e, t) > 0$, a contradiction.

Claim 2: If $\{x, y\}$ is a 2-cut, then $xy \notin E(G)$.

Suppose $xy \in E(G)$ and let $B_1, \ldots, B_m$ be the $\{x, y\}$-bridges of G. By Proposition 2.2,

$$Q(G,t) = \frac{Q(B_1,t)Q(B_2,t)\cdots Q(B_m,t)}{t^{m-1}(t-1)^{m-1}}.$$

Since G is a smallest counterexample and $\mathcal{G}_3$ is splitting-closed, $Q(B_i, t) > 0$ for $i \in [m]$. A contradiction follows.

Claim 3: If $\{x, y\}$ is a 2-cut, then there are precisely three $\{x, y\}$-bridges.

Again let $B_1, \ldots, B_m$ be the $\{x, y\}$-bridges of G. If m is even, then

$$\begin{aligned} Q(G,t) &= Q(G + xy, t) - Q(G/xy, t) \\ &= \frac{Q(B_1 + xy, t)\cdots Q(B_m + xy, t)}{t^{m-1}(t-1)^{m-1}} + \frac{Q(B_1/xy, t)\cdots Q(B_m/xy, t)}{t^{m-1}}. \end{aligned}$$

Since G is a smallest counterexample and $\mathcal{G}_3$ is splitting-closed, all terms in the final expression are positive and so $Q(G,t) > 0$, a contradiction. Thus m is odd. If $m \geq 5$, then choose two bridges B_i and B_j for which $T[V(B_i)]$ and $T[V(B_j)]$ are disconnected and form the graphs $B_{i,j}$ and $B_\cup$ as described in Lemma 4.1. Clearly $B_{i,j}$ and $B_\cup$ are non-separable and have a 3-leaf spanning tree. Since $\mathcal{G}_3$ is splitting-closed, the same is true for all $B_k + xy$, B_k/xy, $k \in [m] \setminus \{i, j\}$. Now, because each of these graphs is smaller than G, the conditions of Lemma 4.1 are satisfied and thus $Q(G,t) > 0$, a contradiction.

Claim 4: If $\{x, y\}$ is a 2-cut, then every $\{x, y\}$-bridge is separable.

Let B be an arbitrary $\{x, y\}$-bridge, say $B = B_1$, and suppose for a contradiction that B is non-separable. Since $xy \notin E(G)$, $|V(B)| \geq 4$. We may assume that B contains at most two leaves of T, since if T has three leaves in B, then $G - \{x, y\}$ has at most two components. Relabelling x and y if necessary there are four cases how $T[V(B)]$ may behave:

Case 1: $T[V(B)]$ is connected.

Case 2: $T[V(B)]$ consists of an isolated vertex x and a path starting at y and covering all vertices of $B - x$.

Case 3: $T[V(B)]$ consists of an isolated vertex x and a tree with precisely three leaves, one of which is y, covering all vertices of $B - x$.

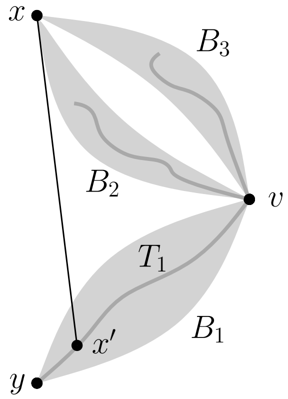

Figure 4: Case 3a of Claim 4.

Case 4: $T[V(B)]$ consists of two disjoint paths P_1 and P_2, starting at x and y respectively, and covering all vertices of B.

In Case 1, $T[V(B)]$ is a 3-leaf spanning tree of B. In Case 2, adding any edge of B incident with x to T also shows that B contains such a spanning tree. Thus in these two cases B, $B + xy$, and B/xy are members of $\mathcal{G}_3$. As G is a smallest counterexample we may apply Lemma 4.1 with $i = 2$ and $j = 3$ which gives $Q(G, t) > 0$, a contradiction.

Two cases remain.

Case 3: $T[V(B)]$ **consists of an isolated vertex** x **and a tree with precisely three leaves, one of which is** y**, covering all vertices of** $B - x$**.**
Let v be the vertex of degree 3 in T and T_1 be the path in T from y to v.

Subcase 3a: $B - x$ **is separable.**
Since $|V(B)| \geq 4$, there is a cut-vertex z of $B - x$. So $\{x, z\}$ is a 2-cut of B and a 2-cut of G. By Claim 3, G has precisely three $\{x, z\}$-bridges, two of which are contained in B. Due to the structure of T, this implies that $v = z$ and T_1 is a Hamiltonian path of the unique $\{z\}$-bridge of $B - x$ which contains y (see Figure 4). Since B is non-separable, $V(T_1) \geq 3$ and x has a neighbour in $V(T_1) \setminus \{y, z\}$. Choose such a neighbour, x', from which the distance to y on T_1 is minimal. It is easy to see that G contains two paths from x to x' avoiding the edge xx' itself. Thus $G - xx'$ is non-separable and has a 3-leaf spanning tree. By Claim 2, G/xx' is also non-separable. Now Proposition 2.1 gives

$$Q(G, t) = Q(G - xx', t) + Q(G/xx', t). \tag{10}$$

Since G is a smallest counterexample, $Q(G - xx', t) > 0$. Thus we have reached a contradiction if $Q(G/xx', t) > 0$. This follows immediately if G/xx' has a 3-leaf spanning tree. Otherwise $|V(B/xx')| \geq 4$, and so we apply Lemma 4.1 to G/xx' with $i = 2$ and $j = 3$. To see that the hypotheses hold we show that all of the graphs B/xx', $B/xx' + xy$ and $B/xx'/xy$ are non-separable and have a 3-leaf spanning tree. That they contain such a spanning tree is clear. Also $B/xx' + xy$ is non-separable since G/xx' is non-separable. Finally, if B/xx' or $B/xx'/xy$ is separable, then $B - \{x, x'\}$ or $B - \{x, x', y\}$ is disconnected. By the choice of x', this implies $\{x', y\}$ is a 2-cut of G. But the structure of T implies that G can only have two $\{x', y\}$-bridges, a contradiction to Claim 3.

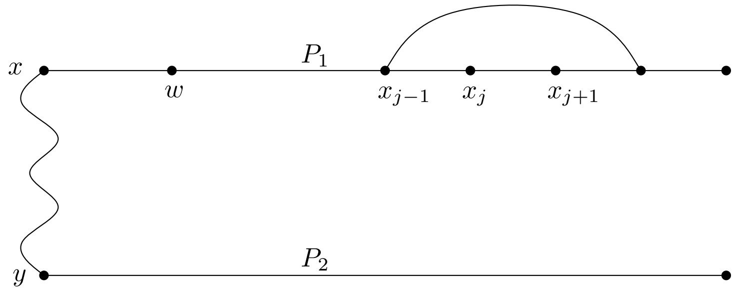

Figure 5: Case 4 of Claim 4, when $w \in V(P_1)$.

Subcase 3b: $B - x$ is non-separable.
Since $|V(B)| \geq 4$, $G - e$ is non-separable for every edge $e \in E(B)$ incident to x. Choose a neighbour, x' of x, such that the distance on $T[V(B)]$ from v to x' is maximal. Since B is non-separable, x has at least two neighbours in B and so $x' \neq v$. By Claim 2 and the above, both $G - xx'$ and G/xx' are non-separable. Furthermore $G - xx'$ has a 3-leaf spanning tree. As before, by (10), we have reached a contradiction if $Q(G/xx') > 0$. This follows immediately if G/xx' has a 3-leaf spanning tree. Otherwise we apply Lemma 4.1 to G/xx' as in Case 3a. The same argument shows that by the choice of x', the hypotheses of Lemma 4.1 hold.

Case 4: $T[V(B)]$ consists of two disjoint paths P_1 and P_2, starting at x and y respectively, and covering all vertices of B.
Let $P_1 = x_1, \ldots, x_{n_1}$ and $P_2 = y_1, \ldots, y_{n_2}$ where $x = x_1$ and $y = y_1$. If x or x_{n_1} has a neighbour x' on P_2, then B contains a 3-leaf spanning tree. As in Cases 1 and 2, this is enough to reach a contradiction. Thus $|V(P_1)| \geq 4$ and all neighbours of x_{n_1} lie on P_1. Apart from its predecessor on P_1, x_{n_1} has at least one other neighbour since B is non-separable. If x_{n_1} has at least two other neighbours, say x_i, x_j with $i < j$, then $G - x_{n_1}x_j$ and $G/x_{n_1}x_j$ are non-separable and have a 3-leaf spanning tree. By Proposition 2.1 we again reach a contradiction. So we may suppose that $d(x_{n_1}) = 2$ and $N(x_{n_1}) = \{x_{n_1-1}, x_i\}$. It follows that $\{x_i, x_{n_1-1}\}$ is a 2-cut of G. Thus, by Claim 3, G has precisely three $\{x_i, x_{n_1-1}\}$-bridges, of which one contains the subpath $P_1[x_i, x_{n_1-1}]$. Call this bridge B_4. Note also that there is some edge e from x_{n_1-1} to the $\{x_i, x_{n_1-1}\}$-bridge of G containing y. Since $T[V(B_4)]$ is a Hamiltonian path of B_4, it follows from Case 1 that B_4 is separable. Thus it has a cut vertex v. Because of the edges e and $x_ix_{n_1}$, we see that $G - \{x_i, v\}$ and $G - \{x_{n_1-1}, v\}$ both have at most two components. From Claim 3 it follows that neither of $\{x_i, v\}$ and $\{x_{n_1-1}, v\}$ are cut-sets of G. Thus B_4 is a path of length 2, $i = n_1 - 3$, and $d(x_{n_1-2}) = 2$. We conclude that there is at least one vertex of degree 2 on the interior of P_1.

Now let $x_j \in V(P_1) \setminus \{x_1, x_{n_1}\}$ be a vertex of degree 2 with j as small as possible (see Figure 5). Then $\{x_{j-1}, x_{j+1}\}$ is a 2-cut and G contains precisely three $\{x_{j-1}, x_{j+1}\}$-bridges. Since each of x_{j-1} and x_{j+1} has a neighbour in each $\{x_{j-1}, x_{j+1}\}$-bridge, there is some edge e from x_{j-1} to one of $x_{j+2}, x_{j+3}, \ldots, x_{n_1}$. Now consider the $\{x_{j-1}, x_{j+1}\}$-bridge, B_y, containing y. This bridge contains a 3-leaf spanning tree covering all of its vertices apart from x_{j+1}. By Case 3, it is separable and has a cut-vertex w. It is easy to see that

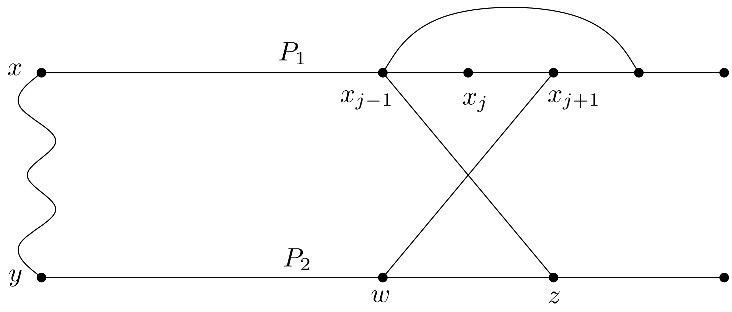

Figure 6: Case 4 of Claim 4, when $w \in V(P_2)$.

w must lie on $P_1[x, x_{j-2}]$ or P_2. Suppose the former (see Figure 5). Since $|V(B_y)| \geq 4$ at least one of $\{x_{j-1}, w\}$ or $\{x_{j+1}, w\}$ is a 2-cut of G. However, because of the edge e and the presence of a path from x to y in another $\{x, y\}$-bridge (indicated in Figure 5), $G - \{x_{j+1}, w\}$ has at most two components. Therefore $\{x_{j-1}, w\}$ is a 2-cut of G and as such gives rise to three $\{x_{j-1}, w\}$-bridges, one of which contains the path $P_1[w, x_{j-1}]$. This path has length at least 2 by Claim 2. By the argument above, $w = x_{j-3}$ and $d(x_{j-2}) = 2$, contradicting the minimality of j.

So we may assume that w lies on P_2 (see Figure 6). Once again at least one of $\{x_{j-1}, w\}$ or $\{x_{j+1}, w\}$ is a 2-cut of G, but $\{x_{j+1}, w\}$ cannot be since $G - \{x_{j+1}, w\}$ has at most two components. It follows that wx_{j+1} is an edge and $\{x_{j-1}, w\}$ is a 2-cut with precisely three bridges. Let B_5 be the $\{x_{j-1}, w\}$-bridge containing the vertex y_{n_2}. So $P_2[w, y_{n_2}]$ is a path of B_5 covering all vertices of B_5 except x_{j-1}. By Case 2, B_5 is separable and has a cut vertex z on $P_2[w, y_{n_2}]$. Because of the edge wx_{j+1}, $G - \{x_{j-1}, z\}$ has at most two components. Therefore $\{x_{j-1}, z\}$ is not a 2-cut of G and zx_{j-1} is an edge.

Finally note that $x_{j-1} \neq x$ and $w \neq y$ or else B would contain a 3-leaf spanning tree. Consider the $\{x_{j-1}, w\}$-bridge B_6 containing x and y. $T[V(B_6)]$ is connected so by Case 1, B_6 is separable and has a cut vertex z'. Since $|V(B_6)| \geq 4$, at least one of $\{x_{j-1}, z'\}$ and $\{w, z'\}$ is a 2-cut of G. However because of the edges zx_{j-1} and wx_{j+1}, both $G - \{x_{j-1}, z'\}$ and $G - \{w, z'\}$ have at most two components. This contradicts Claim 3. □

5 Acknowledgements

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